

Classical stability of BTZ black hole in new massive gravity

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Abstract

We study the stability of the BTZ black hole in the new massive gravity. This is a nontrivial task because the linearized equation around the BTZ black hole background is a fourth order differential equation. Away from the critical point of $m^2\ell^2 = 1/2$, this fourth order equation is split into two second order equations: one describes a massless graviton and the other is designed for a massive graviton, which could be obtained from the Fierz-Pauli action. In this case, calculating quasinormal modes leads to confirm the stability of the BTZ black hole. At the critical point, we derive two left and right logarithmic quasinormal modes from the logarithmic conformal field theory. Finally, we identify two s -massive modes propagating on the black hole background through the conventional black hole stability analysis.

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1 Introduction

It is well known that Einstein gravity in three dimensions has no propagating degrees of freedom. Massive generalizations of three-dimensional gravity allow propagating degrees of freedom. Topologically massive gravity (TMG) is the well-known example obtained by including a gravitational Chern–Simons term with coupling μ [1, 2]. The model was extended by the addition of a cosmological constant term $\Lambda = -1/\ell^2$ to cosmological topologically massive gravity (CTMG) [3]. The gravitational Chern–Simons term is odd under parity and as a result, the theory shows a single massive propagating degree of freedom of a given helicity, whereas the other helicity mode remains massless. The single massive field is realized as a massive scalar $\varphi = z^{3/2}h_{zz}$ when using the Poincare coordinates x^\pm and z covering the AdS_3 spacetimes [4]. However, it was claimed that the negative-energy massive graviton disappears at the critical point of $\mu\ell = 1$ [5]. This cosmological topological massive gravity at the critical point (CCTMG) may be described by the logarithmic conformal field theory (LCFT) [6, 7] even for the zero central charge $c_L = 0$.

Another massive generalization of Einstein gravity in three dimensions was proposed recently by adding a specific quadratic curvature term to the Einstein–Hilbert action [8, 9]. This term was designed to reproduce the ghost-free Fierz–Pauli action for a massive propagating graviton in the linearized approximation. This gravity theory proposed by Bergshoeff, Hohm, and Townsend (BHT) became known as new massive gravity (NMG). Unlike the TMG, the NMG preserves parity. As a result, the gravitons acquire the same mass for both helicity states, indicating two massive propagating degrees of freedom. It was shown that there is no ghost in the linearized BHT gravity by performing a canonical analysis in flat spacetimes [10]. Furthermore, in flat and de Sitter spacetimes, the authors [11] have found two massive propagating scalars σ and ϕ derived from the metric perturbations, satisfying the Klein–Gordon equation $[\nabla^2 - m^2]\{\sigma, \phi\} = 0$. However, up to now, there is no explicit form of two massive scalars propagating on AdS_3 spacetimes, even for the fourth order linearized equation for graviton was known under the transverse and traceless gauge [12].

It is well known that the BTZ black hole as solution to Einstein gravity with Λ is also a black hole solution to the CTMG. However, this does not necessarily imply that there is no difference in the dynamics of perturbations. It is obvious that perturbation discriminates between Einstein gravity and CTMG. Recently, it was shown that the (non-rotating) BTZ black hole is stable for all values of μ against the metric perturbations in the TMG by considering left- and right-moving normal modes [13]. They have confirmed the stability by solving the massive scalar equation of $(\nabla_{\text{BTZ}}^2 - m^2)\varphi = 0$.

In this work, we wish to perform stability analysis on the BTZ black hole in the NMG,

which is a nontrivial task. The basic idea of performing the black hole stability is that one decouples the second order linearized equations and then, manages to arrive at the Schrödinger-type equation with an effective potential for physically propagating fields [14, 15]. If all potentials are positive for whole range outside a event horizon, the black hole under the consideration is stable. If an effective potential is not positive definite everywhere outside a horizon, a special trick called the S -deformation technique may be used to prove the stability [16]. It is well known that a practical tool for testing stability of all kinds of black holes is a numerical investigation of quasinormal frequencies $\omega = \omega_R - i\omega_I$ by imposing the boundary condition: ingoing waves near a event horizon and the Dirichlet boundary condition at infinity [17]. That is, the unstable mode is defined by the condition of

$$\omega_R = 0, \quad \omega_I < 0. \quad (1)$$

in quasinormal mode approach [18]. We wish to perform the stability analysis of the BTZ black holes by computing quasinormal modes of the NMG. However, it seems that this analysis is not a straightforward task in the NMG because the linearized equation is a fourth order differential equation. If the mass parameter m^2 is off the critical value ($m^2 \neq 1/2\ell^2$), the fourth order equation split into two second order equations with transverse and traceless gauge conditions: one is for a massless graviton (gauge artefact) and the other is for a massive graviton, which takes a similar form obtained from the Fierz-Pauli action. However, at the critical point of $m^2 = 1/2\ell^2$ ($c_L = 0, c_R = 0$), the fourth order equation leads to $[\bar{\nabla}^2 - 2\Lambda]^2 h_{\mu\nu} = 0$, which is difficult to be solved unless the LCFT is introduced. Hence, this case is surely beyond the standard stability analysis of a black hole prescribed above. Recently, Sachs and Solodukhin have determined quasinormal mode of black hole spectrum for tensor perturbations in the TMG [19]. In their operator calculation, they have used the chiral highest weight condition of $\bar{L}_1 h_{\mu\nu} = 0$ to derive quasinormal frequencies, which are similar to the scalar quasinormal modes. However, this method is inappropriate to derive quasinormal modes at the critical point $\mu = 1/\ell$ of the TMG. To this end, Sachs has generalized the one-to one correspondence between quasinormal modes in the BTZ black hole and the poles of the retarded correlators in the boundary conformal field theory to include logarithmic operators [20]. On the other hand, Liu and Wang have studied the stability of the BTZ black string against gravitational perturbations in four dimensions, which is very similar to the Fierz-Pauli action in three dimensions [21].

The organization of our work is as follows. In section 2, we briefly review how the BTZ black hole comes out from the NMG. We derive quasinormal modes of the NMG by solving the first order equations with $\Lambda = -1$, which is based on Sachs and Solodukhin method used in the TMG in section 3. However, the left- and right-moving modes are not orthogonal, which

has a problem to be considered as two independent massive modes. Section 4 is focused on deriving quasinormal modes at the critical point of $m^2 = 1/2$ with $\Lambda = -1$. We identify two massive modes Φ and Ψ from the NMG (Fierz-Pauli action) using the conventional black hole stability analysis in section 5. Finally, we discuss similarity and difference in stability between the TMG and the NMG in section 6.

We would like to mention the choice of a cosmological constant Λ . In general, we choose $\Lambda = -1/\ell^2$ except section 3 and 4, where it is chosen to be $\Lambda = -1$ for the simplicity of operator computation.

2 New massive gravity

The NMG action [8] composed of the Einstein-Hilbert action with a cosmological constant λ and higher order curvature terms is given by

$$S_{NMG}^{(3)} = S_{EH}^{(3)} + S_{HC}^{(3)}, \quad (2)$$

$$S_{EH}^{(3)} = \frac{1}{16\pi G} \int d^3x \sqrt{-g} (R - 2\lambda), \quad (3)$$

$$S_{HC}^{(3)} = -\frac{1}{16\pi G m^2} \int d^3x \sqrt{-g} \left(R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2 \right), \quad (4)$$

where G is a three-dimensional Newton's constant and m^2 a parameter with mass dimension 2. From now on, we set $G = 1/8$ for simplicity. The Einstein equation is given by

$$G_{\mu\nu} + \lambda g_{\mu\nu} - \frac{1}{2m^2} K_{\mu\nu} = 0, \quad (5)$$

where the Einstein tensor $G_{\mu\nu}$ and $K_{\mu\nu}$ tensor are given by

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \\ K_{\mu\nu} &= 2\nabla^2 R_{\mu\nu} - \frac{1}{2} \nabla_\mu \nabla_\nu R - \frac{1}{2} \nabla^2 R g_{\mu\nu} \\ &\quad + 4R_{\mu\rho\nu\sigma} R^{\rho\sigma} - \frac{3}{2} R R_{\mu\nu} - R_{\rho\sigma} R^{\rho\sigma} g_{\mu\nu} + \frac{3}{8} R^2 g_{\mu\nu}. \end{aligned} \quad (6)$$

In order to have a black hole solution with dynamical exponent z [22, 23], it is convenient to introduce dimensionless parameters

$$y = m^2 \ell^2, \quad w = \lambda \ell^2, \quad (7)$$

where y and w are proposed to take

$$y = -\frac{z^2 - 3z + 1}{2}, \quad w = -\frac{z^2 + z + 1}{2}. \quad (8)$$

For $z = 1$ (nonrotating) BTZ black hole, one has $y = \frac{1}{2}$ and $w = -\frac{3}{2}$, while $y = -\frac{1}{2}$ and $w = -\frac{13}{2}$ are chosen for $z = 3$ Lifshitz black hole.

In this work, we consider the BTZ black hole solution only

$$ds_{\text{BTZ}}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = - \left(-\mathcal{M} + \frac{r^2}{\ell^2} \right) dt^2 + \frac{dr^2}{\left(-\mathcal{M} + \frac{r^2}{\ell^2} \right)} + r^2 d\phi^2, \quad (9)$$

where \mathcal{M} is the ADM mass determined to be $\mathcal{M} = \frac{r_+^2}{\ell^2}$ with r_+ the horizon radius. Importantly, the mass parameter m^2 and cosmological parameter λ are fixed as

$$m^2 = \frac{1}{2\ell^2}, \quad \lambda = -\frac{3}{2\ell^2} \quad (10)$$

to obtain the BTZ black hole. In this background, taking into account $\bar{g}^{\mu\nu} K_{\mu\nu} = 2m^2\lambda$, the trace of (5) leads to the constant curvature scalar as

$$\bar{R} = 4\lambda = -\frac{6}{\ell^2}, \quad (11)$$

which is the same form as in the Einstein gravity ($R = 6\Lambda$) with the cosmological constant $\Lambda = -1/\ell^2$. On the other hand, the Ricci tensor takes the form

$$\bar{R}_{\mu\nu} = \lambda \bar{g}_{\mu\nu} + \frac{1}{2m^2} K_{\mu\nu} = \frac{4}{3} \lambda \bar{g}_{\mu\nu}, \quad (12)$$

which is the same as that of the Einstein gravity

$$\bar{R}_{\mu\nu} = 2\Lambda \bar{g}_{\mu\nu}. \quad (13)$$

The curvature tensor $\bar{R}_{\mu\rho\nu\sigma}$ takes the form

$$\bar{R}_{\mu\rho\nu\sigma} = \Lambda \left(\bar{g}_{\mu\nu} \bar{g}_{\rho\sigma} - \bar{g}_{\mu\sigma} \bar{g}_{\rho\nu} \right). \quad (14)$$

For an AdS-sized black hole with $r_+ = \ell$, one chooses the unit mass of $\mathcal{M} = 1$, which is designed for finding its quasinormal modes (frequencies). Setting $\ell^2 = 1$ ($\Lambda = -1$), we rewrite the line element (9) in global coordinates:

$$ds_{\text{M}=1}^2 = -\sinh^2(\rho) d\tau^2 + \cosh^2(\rho) d\phi^2 + d\rho^2, \quad (15)$$

where the event horizon is located at $\rho = 0$ ($r_+ = 1$) while the infinity is at $\rho = \infty$ ($r = \infty$). The black hole temperature is $T_H = 1/4\pi$. The metric tensor $\bar{g}_{\mu\nu}$ can be when using the light cone coordinates $u/v = \tau \pm \phi$

$$\bar{g}_{\mu\nu} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \cosh(2\rho) & 0 \\ -\frac{1}{4} \cosh(2\rho) & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (16)$$

Then the metric tensor (16) admits the Killing vector fields L_k , $k = 0, -1, 1$ for local $\text{SL}(2, R) \times \text{SL}(2, R)$ algebra as

$$L_0 = -\partial_u, \quad L_{-1/1} = e^{\mp u} \left[-\frac{\cosh(2\rho)}{\sinh(2\rho)} \partial_u - \frac{1}{\sinh(2\rho)} \partial_v \mp \frac{1}{2} \partial_\rho \right], \quad (17)$$

and $\bar{L}_0, \bar{L}_{-1/1}$ similarly by substituting $u \leftrightarrow v$. Locally, they form a basis of the $\text{SL}(2, R)$ Lie algebra as

$$[L_0, L_{\pm 1}] = \mp L_{\pm 1}, \quad [L_1, L_{-1}] = 2L_0, \quad (18)$$

which are useful for generating the whole tower of quasinormal modes.

3 Quasinormal modes

Considering the perturbation $h_{\mu\nu}$ around the BTZ black hole background $\bar{g}_{\mu\nu}$ in (16)

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad (19)$$

the linearized equation to (5) takes the form

$$\delta G_{\mu\nu}(h) + \lambda h_{\mu\nu} - \frac{1}{2m^2} \delta K_{\mu\nu}(h) = 0, \quad (20)$$

where $\delta G_{\mu\nu}(h)$ and $\delta K_{\mu\nu}(h)$ are the linearized Einstein tensor and linearized $K_{\mu\nu}$. We choose the transverse and traceless (TT) gauge to find a massive graviton propagation on the BTZ black hole background as

$$\bar{\nabla}_\mu h^{\mu\nu} = 0, \quad h \equiv h_\rho{}^\rho = 0. \quad (21)$$

We wish to mention that this covariant gauge is also convenient for studying the conventional stability of black holes in section 5. Then, the fourth order equation (20) is split into

$$\left[\bar{\nabla}^2 - 2\Lambda \right] \left[\bar{\nabla}^2 - \left(m^2 + \frac{5\Lambda}{2} \right) \right] h_{\mu\nu} = 0, \quad (22)$$

which implies two branches of solutions [12]. In this section we choose $\Lambda = -1$ for simplicity of operator calculation. The first equation is

$$\left[\bar{\nabla}^2 - 2\Lambda \right] h_{\mu\nu}^{L/R} = 0, \quad (23)$$

whose solution corresponds to the unphysical modes of left- and right-moving massless gravitons, while the second is the equation describing a physically massive graviton with 2 DOF

$$\left[\bar{\nabla}^2 - \left(m^2 + \frac{5\Lambda}{2} \right) \right] h_{\mu\nu}^M = 0. \quad (24)$$

In order to have a non-negative mass, one requires $m^2 \geq 1/2$. For $m^2 = 1/2$ (the BTZ black hole background), the fourth order equation (22) could not be split into the two second order equations (23) and (24).

First of all, we consider the AdS₃ background [or $M = -1$ in (9)]

$$ds_{\text{AdS}_3}^2 = -\cosh^2(\rho)d\tau^2 + \sinh^2(\rho)d\phi^2 + d\rho^2, \quad (25)$$

one can easily find the solution [5] to (24) as

$$h_{\mu\nu}^{M, \text{AdS}_3} = e^{-ihu - i\bar{h}v} \frac{\sinh^2(\rho)}{(\cosh(\rho))^{h+\bar{h}}} \begin{pmatrix} 1 & \frac{h-\bar{h}}{2} & ia \\ \frac{h-\bar{h}}{2} & 1 & ib \\ ia & ib & -a^2 \end{pmatrix}, \quad (26)$$

where a and b are given by

$$a = \frac{2}{\sinh(2\rho)}, \quad b = \frac{h - \bar{h}}{\sinh(2\rho)}. \quad (27)$$

We have two sets for h and \bar{h} as primary states

$$h + \bar{h} = \frac{2 + \sqrt{2 + 4m^2}}{2}, \quad h - \bar{h} = \pm 2, \quad (28)$$

where \pm denote the solution to the first order equations $(D^{M/\tilde{M}}h)_{\mu\nu} = 0$ with the operators in Eq. (30), respectively. This implies that the solutions to the first order equations are also those to the second order equation. In addition, the solution to (23) in the AdS₃ background (25) has the same form as in (26) when substituting $(h, \bar{h}) = (2, 0)$ for left-moving mode and $(0, 2)$ for right-moving mode: these are also the solutions to $(D^{L/R}h^{L/R})_{\mu\nu} = 0$, respectively.

In this sense, hereafter, we use mainly the first order equations instead of higher order equations to find quasinormal modes. The fourth order equation (22) can be expressed

$$\left(D^L D^R D^M D^{\tilde{M}} h \right)_{\mu\nu} = 0 \quad (29)$$

in terms of mutually commuting operators as

$$(D^{L/R})_\mu^\beta = \delta_\mu^\beta \pm \epsilon_\mu^{\alpha\beta} \bar{\nabla}_\alpha, \quad (D^{M/\tilde{M}})_\mu^\beta = \delta_\mu^\beta \pm \frac{1}{M} \epsilon_\mu^{\alpha\beta} \bar{\nabla}_\alpha \quad (30)$$

with

$$M = \sqrt{m^2 + \frac{1}{2}}. \quad (31)$$

At the critical point (the BTZ black hole background: $m^2 = 1/2$, $M = 1$), the operators D^M and D^L degenerate, while the operators $D^{\tilde{M}}$ and D^R degenerate. Thus, we have to treat it separately in the next section.

On the other hand, the second order massive equation (24) can be expressed as [24]

$$(D^M D^{\tilde{M}} h)_{\mu\nu} = 0. \quad (32)$$

In order to find quasinormal modes, we have to solve (32) together with the TT gauge (21). This approach was used to derive quasinormal modes in the TMG [19], which shows the quasinormal modes of a minimally coupled scalar. We remind the reader that the solution to (32) may be a linear combination of two solutions to the first order equations

$$(D^M h)_{\mu\nu} = 0 \rightarrow \epsilon_\mu^{\alpha\beta} \bar{\nabla}_\alpha h_{\beta\nu} + M h_{\mu\nu} = 0, \quad (33)$$

$$(D^{\tilde{M}} h)_{\mu\nu} = 0 \rightarrow \epsilon_\mu^{\alpha\beta} \bar{\nabla}_\alpha h_{\beta\nu} - M h_{\mu\nu} = 0 \quad (34)$$

together with the TT gauge (21). Let us first find the right-moving solution to (33). The least damped ($n = 0$) quasinormal mode to (33) could be found by considering the form

$$h_{\mu\nu}^M = e^{-i(\omega\tau + k\phi)} \psi_{\mu\nu}^M(\rho) = e^{-ip_+ u - ip_- v} \psi_{\mu\nu}^M(\rho), \quad p_+ \pm p_- = \omega/k, \quad (35)$$

where

$$\psi_{\mu\nu}^M(\rho) = F^M(\rho) \begin{pmatrix} 1 & 0 & \frac{2}{\sinh(2\rho)} \\ 0 & 0 & 0 \\ \frac{2}{\sinh(2\rho)} & 0 & \frac{4}{\sinh^2(2\rho)} \end{pmatrix}. \quad (36)$$

The transversality condition of $\bar{\nabla}_\mu h^{\mu\nu} = 0$ implies the chiral highest weight condition of $\bar{L}_1 h_{\mu\nu} = 0$ under the form of $h_{\mu\nu}^M$ in (35), giving the constraint

$$\left[2ip_+ + 2ip_- \cosh(2\rho) + \sinh(2\rho) \partial_\rho \right] F^M(\rho) = 0, \quad (37)$$

whose solution is given by

$$F^M(\rho) = [\sinh(\rho)]^{-2ip_-} [\tanh(\rho)]^{-ik}. \quad (38)$$

From (33) for $\mu = \nu = \rho$, p_- is determined to be

$$p_- = -ih_R(M), \quad h_R(M) = -\frac{M}{2} - \frac{1}{2}. \quad (39)$$

At the first sight, the $n = 0$ quasinormal mode seems to be

$$h_{\mu\nu}^M = e^{-ik(\tau + \phi) - 2h_R(M)\tau} \psi_{\mu\nu}^M(\rho). \quad (40)$$

Considering the form of quasinormal frequency

$$\omega = \omega_R - i\omega_I, \quad \omega_I > 0 \quad (41)$$

we read off it from (40)

$$\omega_M = k - 2ih_R(M). \quad (42)$$

We note here that “ k ” is the angular quantum number as well as the real part of quasinormal frequency. In order that ω_M be a quasinormal mode, $h_R(M)$ is required to be positive and thus, one has to have $M < -1$. However, it is impossible to make $M < -1$ because $M \geq 1$, as is shown in (31). Hence it is clear that (40) is not considered as a right-moving quasinormal mode for the least damped case.

The next step is to find the right-moving solution to (34). This solution is obtained simply by replacing M by $\tilde{M} = -M$ [equivalently, $h_R(M)$ by $h_R(\tilde{M})$] in (40) as

$$h_{\mu\nu}^{\tilde{M}} = e^{-ik(\tau+\phi)-2h_R(\tilde{M})\tau} \psi_{\mu\nu}^{\tilde{M}}(\rho), \quad (43)$$

where

$$h_R(\tilde{M}) = \frac{M}{2} - \frac{1}{2}, \quad M \geq 1. \quad (44)$$

It seems that (43) is regarded as a right-moving quasinormal mode of the least damped case whose quasinormal frequency is given by

$$\omega_{\tilde{M}} = k - 2ih_R(\tilde{M}). \quad (45)$$

Hence, we could construct the overtone quasinormal modes for this solution. Introducing the operator combination $L_{-1}\bar{L}_{-1}$ which replaces ω_I by $\omega_I - 2$, the n^{th} -overtone quasinormal mode is constructed by

$$h_{\mu\nu}^{\tilde{M},n} = (L_{-1}\bar{L}_{-1})^n h_{\mu\nu}^{\tilde{M}} \quad (46)$$

whose quasinormal frequency takes the form

$$\omega_{\tilde{M}}^n = k - 2i[h_R(\tilde{M}) + n], \quad n \in \mathbb{Z}. \quad (47)$$

Similarly, when imposing the anti-chiral highest weight condition of $L_1 h_{\mu\nu}^M = 0$ and thus, $p_+ = -ih_L(M)$, the left-moving quasinormal modes to (33) take the form

$$h_{\mu\nu}^{M,n} = (L_{-1}\bar{L}_{-1})^n h_{\mu\nu}^M = e^{ik(\tau-\phi)-2h_L(M)\tau} (L_{-1}\bar{L}_{-1})^n \psi_{\mu\nu}^M(\rho), \quad (48)$$

where

$$\psi_{\mu\nu}^M(\rho) = [\sinh(\rho)]^{-2h_L(M)} [\tanh(\rho)]^{ik} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \frac{2}{\sinh(2\rho)} \\ 0 & \frac{2}{\sinh(2\rho)} & \frac{4}{\sinh^2(2\rho)} \end{pmatrix}, \quad h_L(M) = \frac{M}{2} - \frac{1}{2}, \quad (49)$$

whose quasinormal modes are given by

$$\omega_M^n = -k - 2i \left[h_L(M) + n \right], \quad n \in \mathbb{Z}. \quad (50)$$

We note that two quasinormal modes $h_{\mu\nu}^{\tilde{M}}$ and $h_{\mu\nu}^M$ are ingoing (the right- and left-moving) modes at the horizon $\rho = 0$ and are normalizable modes at $\rho = \infty$ satisfying the Dirichlet boundary condition. It is worth noting that the left-moving quasinormal modes to (34) with

$$h_L(\tilde{M}) = -\frac{M}{2} - \frac{1}{2} \quad (51)$$

is not available since $h_L(\tilde{M}) < 0$ due to $M \geq 1$. We summarize the possible quasinormal modes in the following Table. In the BTZ black hole background, (33) allows the L-mode

| equation | right-moving (R) mode | left-moving (L) mode |
|----------|--------------------------------|--------------------------------|
| (33) | $h_{\mu\nu}^{M,R}$ (X) | $h_{\mu\nu}^{M,L}$ (O) |
| (34) | $h_{\mu\nu}^{\tilde{M},R}$ (O) | $h_{\mu\nu}^{\tilde{M},L}$ (X) |

only, while (34) does the R-mode.

In this section, we have obtained the L/R-moving quasinormal modes, which show at least the s -mode stability using (1): for $k = 0$ ($\omega_R = 0$), one has $\omega_{I,\tilde{M}/M}^n = 2n + h_{R/L}(\tilde{M}/M) > 0$. In this case, $k = 0$ requires the disappearance of the L/R-moving modes.

At this stage, it seems appropriate to comment on the similarity and difference between TMG and NMG. In deriving the quasinormal modes, we have used the operator method. As far as this method is concerned, there is no essential difference between TMG and NMG. The difference is that the substitution of $M \rightarrow \mu$ in (33) is necessary for the TMG and (34) is required additionally for the NMG because the parity is preserved in the NMG. However, we note that the L/R-moving modes (46) and (48) are not orthogonal to each other even s -mode ($k = 0$) in the NMG because they contain $h_{\rho\rho}$ commonly. This gives rise to a significant difference in obtaining the quasinormal modes of the same BTZ black hole between two massive gravity theories.

Finally, at the chiral point of $M = 1$, two $h_{\mu\nu}^{\tilde{M}}$ and $h_{\mu\nu}^M$ are no longer $n = 0$ quasinormal modes because of $\omega_I = 0$ ($h_{L/R}(1) = 0$). Hence, we will treat it in the next section. In addition, we mention that requiring both of $\bar{L}_{-1}h_{\mu\nu} = 0$ and $L_{-1}h_{\mu\nu} = 0$ lead to normal modes with ingoing and outgoing fluxes at the event horizon as in Ref. [13], which is not the condition for obtaining quasinormal modes.

4 Quasinormal modes at the critical point

In order to see what happens in the perturbation at the critical point (the BTZ black hole background), we need to solve the fourth order linearized equation. In this section we choose $\Lambda = -1$ for simplicity of operator computation. In general, a mode annihilated by $D^M(D^L)[D^R]\{(D^L)^2$ but not by $D^L\}$ is called massive (left-moving) [right-moving] {left-logarithmic (L,new)} and is denoted by $h_{\mu\nu}^M(h_{\mu\nu}^L)[h_{\mu\nu}^R]\{h_{\mu\nu}^{L,new}\}$ [25]. Also, a mode annihilated by $(D^R)^2$ but not by D^R is called right-logarithmic mode and is denoted by $h_{\mu\nu}^{R,new}$. Away from the critical point ($m^2 \neq 1/2$), the general solution to (29) is obtained by linearly combining left, right, and two massive modes.

At the critical point, $m^2 = 1/2$, D^M degenerates into D^L , while $D^{\tilde{M}}$ degenerates into D^R . The L/R-moving modes are purely gauge degrees of freedom (unphysical modes), whereas two massive modes and their logarithmic partners constitute physically propagating bulk modes. At the critical point (the BTZ black hole background), the fourth order equation (22) becomes [26]

$$(D^R D^L)^2 h_{\mu\nu}^{L/R,new} = 0 \rightarrow [\bar{\nabla}^2 - 2\Lambda]^2 h_{\mu\nu}^{L/R,new} = 0, \quad (52)$$

which is basically different from the second order equations (23) and (24). The reason is clear when observing

$$(D^R D^L h_{\mu\nu}^{L/R,new})_{\mu\nu} = -2h_{\mu\nu}^{L/R} \rightarrow [\bar{\nabla}^2 - 2\Lambda] h_{\mu\nu}^{L/R,new} = 2h_{\mu\nu}^{L/R}, \quad (53)$$

which was derived in Appendix I in detail. See Ref. [26] for the derivation this relation on AdS_3 background. This naturally leads to (52) when operating $(\bar{\nabla}^2 - 2\Lambda)$ on both sides and using (23). In this case, considering $M \rightarrow L$ and $\tilde{M} \rightarrow R$, one has

$$\begin{aligned} h_{\mu\nu}^L &= e^{ik(\tau-\phi)} \psi_{\mu\nu}^L(\rho) \\ &= e^{ik(\tau-\phi)} [\tanh(\rho)]^{ik} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \frac{2}{\sinh(2\rho)} \\ 0 & \frac{2}{\sinh(2\rho)} & \frac{4}{\sinh^2(2\rho)} \end{pmatrix}, \end{aligned} \quad (54)$$

$$\begin{aligned} h_{\mu\nu}^R &= e^{-ik(\tau+\phi)} \psi_{\mu\nu}^R(\rho) \\ &= e^{-ik(\tau+\phi)} [\tanh(\rho)]^{-ik} \begin{pmatrix} 1 & 0 & \frac{2}{\sinh(2\rho)} \\ 0 & 0 & 0 \\ \frac{2}{\sinh(2\rho)} & 0 & \frac{4}{\sinh^2(2\rho)} \end{pmatrix}. \end{aligned} \quad (55)$$

Then, we construct two newly logarithmic solutions of L/R modes to (52)

$$h_{\mu\nu}^{L,new} = \partial_{m^2} h_{\mu\nu}^M|_{m^2=1/2} = y(\tau, \rho) h_{\mu\nu}^L, \quad (56)$$

$$h_{\mu\nu}^{R,new} = \partial_{m^2} h_{\mu\nu}^{\tilde{M}}|_{m^2=1/2} = y(\tau, \rho) h_{\mu\nu}^R, \quad (57)$$

where $y(\tau, \rho)$ is defined by

$$y(\tau, \rho) = -\tau - \ln[\sinh(\rho)]. \quad (58)$$

We are now in a position to construct two quasinormal modes in the BTZ black hole background inspired by left-logarithmic sector of TMG. Unfortunately, $h_{\mu\nu}^{L,new}$ is growing in time τ and ρ , showing disqualification as a quasinormal mode. Hence, Sachs [20] has proposed that the left-logarithmic quasinormal mode can be constructed by considering

$$h_{\mu\nu}^{L,(n)} = \left[\bar{L}_{-1} L_{-1} \right]^n h_{\mu\nu}^{L,new}. \quad (59)$$

For example, a candidate for quasinormal mode is the first descendent given by

$$\begin{aligned} h_{\mu\nu}^{L,(1)} &= (\bar{L}_{-1} L_{-1}) h_{\mu\nu}^{L,new} \\ &= \left[\frac{1}{2} - y(\tau, \rho) + \frac{ik}{2} y(\tau, \rho) \right] \psi_{\mu\nu}^{L,(1)} \\ &+ \frac{2e^{-2\tau}}{\sinh^2(\rho)} \left[\cosh^2(\rho) + \frac{1}{2} k^2 y(\tau, \rho) + ik \left(y(\tau, \rho) - \frac{1 + \cosh^2(\rho)}{2} \right) \right] h_{\mu\nu}^L, \end{aligned} \quad (60)$$

as the left-logarithmic solution. Here

$$\psi_{\mu\nu}^{L,(1)} = \frac{2e^{-2\tau}}{\sinh^2(\rho)} e^{ik(\tau-\phi)} (\tanh(\rho))^{ik} \begin{pmatrix} 0 & 1 & \frac{2}{\sinh(2\rho)} \\ 1 & 1 & \frac{2 \cosh(\rho)}{\sinh(\rho)} \\ \frac{2}{\sinh(2\rho)} & \frac{2 \cosh(\rho)}{\sinh(\rho)} & \frac{4(1+2 \cosh(2\rho))}{\sinh^2(2\rho)} \end{pmatrix}. \quad (61)$$

where we observe that $h_{\mu\nu}^{L,(1)}$ shows a genuine quasinormal mode with exponential fall-off in τ and ρ . Note that in the limit of $k \rightarrow 0$, the first descendent becomes (2.18) of Ref. [20] for the TMG without $\psi_{\mu\nu}^L$ which may be considered as pure gauge*. Here we emphasize again that the $k \rightarrow 0$ limit denotes both the s -mode of ϕ and $\omega_R = 0$, which confirms the s -left logarithmic mode stability of BTZ black hole. The higher descendent modes in principle can be obtained by using (59), however, it seems to be a formidable task. Thus, we confine ourselves to the $k = 0$ (s -mode) case, for simplicity, and obtain the second descendent mode as

$$h_{\mu\nu}^{L,(2)} = \left[\bar{L}_{-1} L_{-1} \right]^2 h_{\mu\nu}^{L,new} = \frac{e^{-4\tau}}{\sinh^4 \rho} \left[\psi_{\mu\nu}^{L,(2A)} - 12y(\tau, \rho) \psi_{\mu\nu}^{L,(2B)} \right], \quad (62)$$

*We thank I. Sachs for pointing out it.

where

$$\begin{aligned}\psi_{\mu\nu}^{L,(2A)} &= \begin{pmatrix} 6 & 11 + 9 \cosh(2\rho) & \frac{2(11+15 \cosh(2\rho))}{\sinh(2\rho)} \\ 11 + 9 \cosh(2\rho) & 11 + 9 \cosh(2\rho) & 4(7 + 3 \cosh(2\rho)) \coth(\rho) \\ \frac{2(11+15 \cosh(2\rho))}{\sinh(2\rho)} & 4(7 + 3 \cosh(2\rho)) \coth(\rho) & \frac{4(26+31 \cosh(2\rho)+9 \cosh(4\rho))}{\sinh^2(2\rho)} \end{pmatrix}, \\ \psi_{\mu\nu}^{L,(2B)} &= \begin{pmatrix} 1 & 2 + \cosh(2\rho) & 4 \coth(\rho) \\ 2 + \cosh(2\rho) & 2 + \cosh(2\rho) & (5 + \cosh(2\rho)) \coth(\rho) \\ 4 \coth(\rho) & (5 + \cosh(2\rho)) \coth(\rho) & \frac{2(3+2 \cosh(2\rho))}{\sinh^2(\rho)} \end{pmatrix}. \end{aligned} \quad (63)$$

Restoring the k -dependence, we may find by induction that the higher modes $h_{\mu\nu}^{R,(n)}$ ($n \geq 2$) are constructed by replacing $e^{-2\tau}$ by $e^{-2n\tau}$. Then, the quasinormal frequencies may be given by

$$\omega_L^n = -k - 2in, \quad n \in Z, \quad (64)$$

which are the same quasinormal frequencies of left-moving modes ω_M^n with $h_R(\tilde{M} = 1) = 0$ in (47).

The right-logarithmic quasinormal modes could be similarly constructed by

$$\begin{aligned}h_{\mu\nu}^{R,(1)} &= (\bar{L}_{-1}L_{-1})h_{\mu\nu}^{R,new} \\ &= \left[\frac{1}{2} - y(\tau, \rho) - \frac{ik}{2}y(\tau, \rho) \right] \psi_{\mu\nu}^{R,(1)} \\ &+ \frac{2e^{-2\tau}}{\sinh^2(\rho)} \left[\cosh^2(\rho) + \frac{1}{2}k^2y(\tau, \rho) - ik \left(y(\tau, \rho) - \frac{1 + \cosh^2(\rho)}{2} \right) \right] h_{\mu\nu}^R, \end{aligned} \quad (65)$$

where

$$\psi_{\mu\nu}^{R,(1)} = \frac{2e^{-2\tau}}{\sinh^2(\rho)} e^{-ik(\tau+\phi)} (\tanh(\rho))^{-ik} \begin{pmatrix} 1 & 1 & \frac{2 \cosh(\rho)}{\sinh(\rho)} \\ 1 & 0 & \frac{2}{\sinh(2\rho)} \\ \frac{2 \cosh(\rho)}{\sinh(\rho)} & \frac{2}{\sinh(2\rho)} & \frac{4(1+2 \cosh(2\rho))}{\sinh^2(2\rho)} \end{pmatrix}. \quad (66)$$

Here we observe that $h_{\mu\nu}^{R,(1)}$ is a genuine quasinormal mode with exponential fall-off in τ and ρ . Considering the $k = 0$ (s -mode) case, for simplicity, the second descendent mode is given by

$$h_{\mu\nu}^{R,(2)} = [\bar{L}_{-1}L_{-1}]^2 h_{\mu\nu}^{R,new} = \frac{e^{-4\tau}}{\sinh^4 \rho} \left[2\psi_{\mu\nu}^{R,(2A)} - 12y(\tau, \rho)\psi_{\mu\nu}^{R,(2B)} \right], \quad (67)$$

where

$$\begin{aligned}\psi_{\mu\nu}^{R,(2A)} &= \begin{pmatrix} 10 + 7 \cosh(2\rho) & 9 + 3 \cosh(\rho) + 5 \cosh(2\rho) & \frac{31+33 \cosh(2\rho)+4 \cosh(4\rho)}{\sinh(2\rho)} \\ 9 + 3 \cosh(\rho) + 5 \cosh(2\rho) & 6 & \frac{21+25 \cosh(2\rho)}{\sinh(2\rho)} \\ \frac{31+33 \cosh(2\rho)+4 \cosh(4\rho)}{\sinh(2\rho)} & \frac{21+25 \cosh(2\rho)}{\sinh(2\rho)} & \frac{2P(\rho)}{\sinh^2(2\rho)} \end{pmatrix}, \\ \psi_{\mu\nu}^{R,(2B)} &= \begin{pmatrix} 2 + \cosh(2\rho) & 2 + \cosh(2\rho) & (5 + \cosh(2\rho)) \coth(\rho) \\ 2 + \cosh(2\rho) & 1 & 4 \coth(\rho) \\ (5 + \cosh(2\rho)) \coth(\rho) & 4 \coth(\rho) & \frac{2(3+2 \cosh(2\rho))}{\sinh^2(\rho)} \end{pmatrix}\end{aligned}\quad (68)$$

with

$$P(\rho) = 36 + 4 \cosh(\rho) + 40 \cosh(2\rho) + 10 \cosh(4\rho) + 6 \sinh(2\rho) + 3 \sinh(4\rho). \quad (69)$$

Assuming the k -dependence again, the higher modes $h_{\mu\nu}^{R,(n)}$ ($n \geq 2$) can be obtained by replacing $e^{-2\tau}$ by $e^{-2n\tau}$ as well. Thus, the quasinormal frequencies may be given by

$$\omega_L^n = k - 2in, \quad n \in Z, \quad (70)$$

which are the same quasinormal frequencies of right-moving modes ω_M^n with $h_L(M=1)=0$ in (50).

In this section, we have obtained L/R-logarithmic quasinormal modes, which show at least the s -mode stability using (1): for $k=0$ ($\omega_R=0$), one has $\omega_{I,L/R}^n = 2n > 0$.

Finally, we note the similarity and difference between TMG and NMG. In deriving the quasinormal modes at the critical point, we have used the operator method. There is no essential difference between TMG and NMG. The difference is that the L-logarithmic mode approach is necessary for the TMG, while both of L/R logarithmic approaches are required for the NMG because the parity is preserved in the NMG. However, it is not clear that two L/R-logarithmic quasinormal modes are orthogonal to each other even for the s -mode ($k=0$). This is an important issue because the orthogonality may guarantee two independent massive quasinormal modes at the critical point.

5 Two s -massive modes in NMG

First of all, we point out a few problems on the previous tensor quasinormal modes. Their frequencies were constructed as those of a minimally coupled scalar mode. The L/R modes of (46) and (48) are not orthogonal to each other even for s -mode ($k=1$) because they contain $h_{\rho\rho}$ commonly. Hence, we did not confirm that the L/R-moving modes are two truly propagating modes in the new massive gravity. This asks directly what is the explicit form

of two physically massive modes propagating in the BTZ black hole. In this section, we wish to identify two massive graviton modes propagating in the BTZ black hole background.

In order to perform the stability analysis of massive graviton with 2 DOF, we first recall Eq. (9) as the BTZ black hole background. In this background the perturbation equations (24) can be rewritten as

$$\bar{\nabla}_{\text{BTZ}}^2 h_{\mu\nu} - \left[m^2 + \frac{5\Lambda}{2} \right] h_{\mu\nu} = 0. \quad (71)$$

This equation is similar to the Fierz-Pauli massive equation

$$\bar{\nabla}_{\text{BTZ}}^2 h_{\mu\nu} - \left[m^2 + 2\Lambda \right] h_{\mu\nu} = 0, \quad (72)$$

which is derived from Fierz-Pauli action together with the TT gauge. In order to solve (71) explicitly with the TT gauge, we consider the following two distinct perturbing metric ansatz: the type I has two off-diagonal components h_0 and h_1

$$h_{\mu\nu}^I = \begin{pmatrix} 0 & 0 & h_0(r) \\ 0 & 0 & h_1(r) \\ h_0(r) & h_1(r) & 0 \end{pmatrix} e^{\omega_h t} e^{ik\phi}, \quad (73)$$

while for the type II, the metric tensor takes the form with four components H_0 , H_1 , H_2 , and H_3 as [21]

$$h_{\mu\nu}^{II} = \begin{pmatrix} H_0(r) & H_1(r) & 0 \\ H_1(r) & H_2(r) & 0 \\ 0 & 0 & H_3(r) \end{pmatrix} e^{\omega_h t} e^{ik\phi}. \quad (74)$$

At this stage, we note that the above two choices of (73) and (74) are working only for s -mode ($k = 0$) perturbation. Also, two are orthogonal to each other. In Appendix II, we have shown that $k = 0$ case leads to two consistent decoupling processes for I and II metric perturbations, when solving (71).

Substituting Eq. (73) into Eq. (71) and eliminating h_1 from (t, ϕ) and (r, ϕ) components of (71), we obtain

$$\begin{aligned} & \{ (m^2 - 1/2\ell^2)(r^2/\ell^2 - \mathcal{M}) + \omega_h^2 \} h_0'' + \left\{ \frac{r^2/\ell^2 + \mathcal{M}}{r(r^2/\ell^2 - \mathcal{M})} \omega_h^2 - \frac{r^2/\ell^2 - \mathcal{M}}{r} (m^2 - 1/2\ell^2) \right\} h_0' \\ & - \left\{ \frac{4\omega_h^2}{\ell^2(r^2/\ell^2 - \mathcal{M})} + \left(m^2 - 1/2\ell^2 + \frac{\omega_h^2}{r^2/\ell^2 - \mathcal{M}} \right)^2 \right\} h_0 = 0. \end{aligned} \quad (75)$$

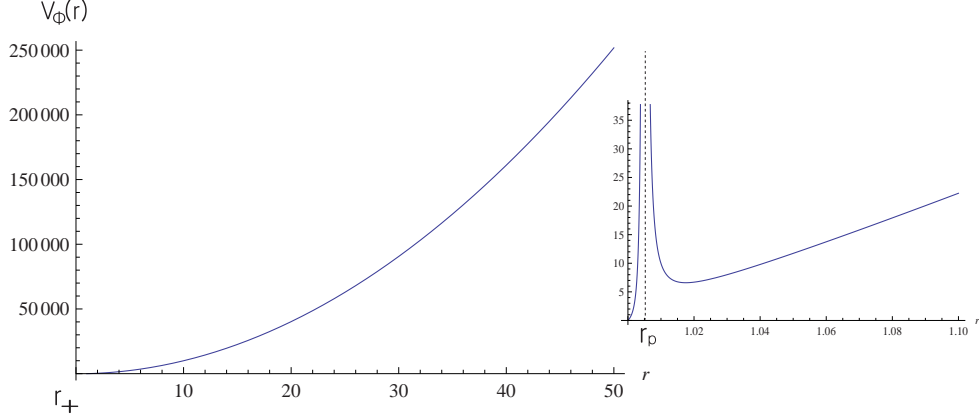


Figure 1: V_Φ graphs as function of r with fixed values $\ell^2 = 1$, $\mathcal{M} = 1$, $\omega = 1$, and $m = 10$. For $m^2 \neq 1/2\ell^2$ case, V_Φ blows up at $r = r_p = \sqrt{l^2\mathcal{M} + \omega^2/(m^2 - 1/2\ell^2)}$ (≈ 1.005 , near the event horizon), while for the critical $m^2 = 1/2\ell^2$ case, there is no the blowing-up point.

Note that we consider only s -mode ($k = 0$) for components $(t, r), (r, r)$ of Eq. (71). At the horizon $r = r_+$ and the spatial infinity $r = \infty$, a solution to the above equation behaves as

$$h_0 \sim (r - r_+)^{\pm \frac{\omega_h}{2\sqrt{\mathcal{M}/\ell^2}}} = e^{\pm \frac{\omega_h}{2\sqrt{\mathcal{M}/\ell^2}} \ln[r - r_+]}, \quad \text{and} \quad h_0 \sim r^{1 \pm \sqrt{\ell^2 m^2 + 1/2}}, \quad (76)$$

where $r_+ = \sqrt{\mathcal{M}\ell^2}$. Introducing the tortoise coordinate r^* in $[dr^* = dr/(-\mathcal{M} + r^2/\ell^2)]$, and redefining $\omega_h = -i\omega$ and a new field Φ as

$$\Phi = \frac{h_0}{g(r)} \quad (77)$$

with

$$g(r) = \sqrt{r \{ (m^2 - 1/2\ell^2)r^2/\ell^2 - (m^2 - 1/2\ell^2)\mathcal{M} + \omega_h^2 \}}, \quad (78)$$

Eq.(75) can be written as the Schrödinger-type equation

$$\frac{d^2\Phi}{dr^{*2}} + [\omega^2 - V_\Phi(\omega, r)]\Phi = 0, \quad (79)$$

where the ω -dependent potential takes the form

$$V_\Phi(\omega, r) = (r^2/\ell^2 - \mathcal{M}) \left[m^2 + \frac{13}{4\ell^2} - \frac{3\mathcal{M}}{4r^2} + \frac{3(m^2 - 1/2\ell^2)^2 r^2 (r^2/\ell^2 - \mathcal{M})}{\ell^4 \{ (m^2 - 1/2\ell^2)(-\mathcal{M} + r^2/\ell^2) - \omega^2 \}^2} \right. \\ \left. + \frac{2(m^2 - 1/2\ell^2)(2\mathcal{M} - 3r^2/\ell^2)}{\ell^2 \{ (m^2 - 1/2\ell^2)(-\mathcal{M} + r^2/\ell^2) - \omega^2 \}} \right]. \quad (80)$$

It is important to note that for $m^2 \geq 1/2\ell^2$, the above potential is always positive for whole range of $r_+ \leq r^* \leq \infty$ even though it blows up at $r = r_p$ (see Fig.1), which implies that the type I-perturbation is stable.

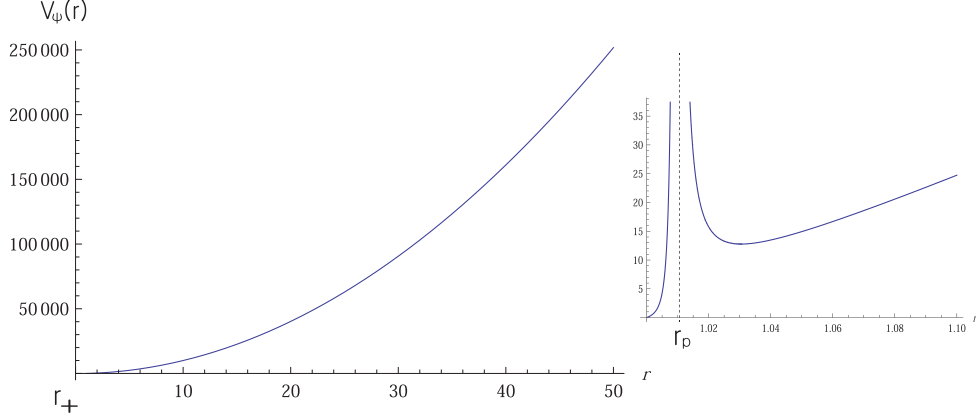


Figure 2: V_Ψ graphs as function of r with fixed values $\ell^2 = 1$, $\mathcal{M} = 1$, $\omega = 1$, and $m = 10$. In right graph, V_Ψ blows up at $r = r_p = \sqrt{\ell^2 \mathcal{M} + (\mathcal{M}/\ell^2 + \omega^2)/(m^2 + 1/2\ell^2)}$ (≈ 1.01 near the even horizon $r = r_+$).

On the other hand, plugging Eq.(74) into Eq.(71) and after manipulations we obtain the second differential equation for $H_1(r)$ as

$$\begin{aligned} & \left\{ (m^2 - 1/2\ell^2)(r^2/\ell^2 - \mathcal{M}) + r^2/\ell^4 - 2\mathcal{M}/\ell^2 + \omega_h^2 \right\} H_1'' + \left\{ \frac{7r^2/\ell^2 - \mathcal{M}}{r(r^2/\ell^2 - \mathcal{M})} \omega_h^2 \right. \\ & + \frac{5r^2/\ell^2 - \mathcal{M}}{r} (m^2 - 1/2\ell^2) + \frac{5r^4/\ell^4 - 13\mathcal{M}r^2/\ell^2 + 2\mathcal{M}^2}{\ell^2 r (r^2/\ell^2 - \mathcal{M})} \left. \right\} H_1' + \left\{ \frac{6r^4/\ell^4 - \mathcal{M}^2}{r^2 (r^2/\ell^2 - \mathcal{M})^2} \omega_h^2 \right. \\ & + \frac{2r^4/\ell^4 - 2\mathcal{M}r^2/\ell^2 - \mathcal{M}^2}{r^2 (r^2/\ell^2 - \mathcal{M})} (m^2 - 1/2\ell^2) + \frac{(3r^2/\ell^2 - 2\mathcal{M})(r^4/\ell^4 - 4\mathcal{M}r^2/\ell^2 - \mathcal{M}^2)}{\ell^2 r^2 (r^2/\ell^2 - \mathcal{M})^2} \\ & \left. - \left(m^2 - 1/2\ell^2 + \frac{\omega_h^2}{r^2/\ell^2 - \mathcal{M}} \right)^2 \right\} H_1 = 0. \end{aligned} \quad (81)$$

In this case, we also consider only s -mode ($k = 0$) for component (t, ϕ) of Eq.(71). We have wave forms near the horizon and spatial infinity

$$H_1 \sim (r - r_+)^{-1 \pm \frac{\omega_h}{2\sqrt{\mathcal{M}/\ell^2}}} = e^{[-1 \pm \frac{\omega_h}{2\sqrt{\mathcal{M}/\ell^2}}] \ln[r - r_+]}, \quad \text{and} \quad H_1 \sim r^{-2 \pm \sqrt{\ell^2 m^2 + 1/2}}. \quad (82)$$

Introducing the tortoise coordinate r^* , and redefining $\omega_h = -i\omega$ and a new field Ψ

$$\Psi = \frac{H_1}{f(r)} \quad (83)$$

with

$$f(r) = \frac{\sqrt{\mathcal{M}(m^2 - 1/2\ell^2) + 2\mathcal{M}/\ell^2 - (m^2 - 1/2\ell^2)r^2/\ell^2 - r^2/\ell^4 - \omega_h^2}}{\sqrt{r(r^2/\ell^2 - \mathcal{M})^2}}, \quad (84)$$

Eq.(81) can be written as the Schrödinger-type equation

$$\frac{d^2\Psi}{dr^{*2}} + [\omega^2 - V_\Psi(\omega, r)]\Psi = 0, \quad (85)$$

where the ω -dependent potential is given by

$$V_\Psi(\omega, r) = (r^2/\ell^2 - \mathcal{M}) \left[m^2 + \frac{5}{4\ell^2} - \frac{3\mathcal{M}}{4r^2} + \frac{3r^2(m^2 + 1/2\ell^2)^2(r^2/\ell^2 - \mathcal{M})}{\ell^4(\mathcal{M}m^2 + 3\mathcal{M}/2\ell^2 - m^2r^2/\ell^2 - r^2/2\ell^4 + \omega^2)^2} \right. \\ \left. + \frac{4(m^2 + 1/2\ell^2)(r^2/\ell^2 - \mathcal{M})}{\ell^2(\mathcal{M}m^2 + 3\mathcal{M}/2\ell^2 - m^2r^2/\ell^2 - r^2/2\ell^4 + \omega^2)} \right]. \quad (86)$$

It is important to note that for $m^2 \geq 1/2\ell^2$, the above potential is always positive for whole range of $r_+ \leq r \leq \infty$ ($-\infty \leq r^* \leq 0$) (see Fig.2) even though it blows up at $r = r_p$, which implies that the type II-perturbation is stable. This has been confirmed by computing quasinormal modes numerically when using (81) in Ref. [21], instead of (85).

Finally, we have shown that the positivity of two potentials leads to the stability condition of the BTZ black hole

$$m^2 \geq \frac{1}{2\ell^2}, \quad (87)$$

which is consistent with the condition of $M \geq \ell$.

6 Discussions

We have performed the stability analysis of the BTZ black hole in the NMG. We have derived quasinormal modes of the NMG by solving the first order equations, which is based on Sachs and Solodukhin method used in the TMG. However, we have observed that the left- and right-modes are not orthogonal, which has a problem to be considered as two independent massive modes. Hence, we have identified two massive modes from the NMG (Fierz-Pauli action) using the conventional black hole stability analysis. Furthermore, we have obtained left- and right logarithmic quasinormal modes at the critical point of $m^2\ell^2 = 1/2$ using the LCFT.

We discuss similarity and difference in stability between the TMG and the NMG. In the case of the TMG, the BTZ black hole is stable for any Chern-Simons coupling constant $\mu > 0$ by computing (33) with replacing M by μ , while the stability of the BTZ black hole is guaranteed for $m^2\ell^2 > 1/2$ ($M > \ell$) by computing two first order equations (33) and (34). In the TMG approach, the authors [13] have confirmed the stability by using that a single massive scalar of $\varphi = z^3 h_{zz}$ satisfies the Klein-Gordon equation in the AdS_3 background. On the other hand, the two left- and right-modes of the NMG [(46) and (48)] are not orthogonal

to each other, implying that the two are not considered as two independent massive modes propagating on the BTZ black hole background. This requires a reanalysis of stability of the BTZ black hole in the conventional black hole approach. We have obtained two propagating s -massive modes Φ and Ψ by solving the Schrödinger equations (79) and (85) directly, which are surely not the Schrödinger equations for a massive minimally coupled scalar.

However, the difference between the NMG and the Fierz-Pauli action is just the presence of the critical point of $m^2\ell^2 = 1/2$ ($c_L = 0, c_R = 0$), whose equation is given by the fourth order linearized equation (52). This point exactly coincides with providing the BTZ black hole solution. Actually, there is no definite way to solve this equation. In the strict sense of stability of the BTZ black hole in the NMG, we have confined ourselves to solving (52) by using the LCFT technique in section 5. However, this method is very restrictive and thus, there is no definite way to confirm its results at present. Particularly, the s -mode ($k = 0$) computation could be easily done to derive the left- and right logarithmic quasinormal modes.

At this stage, we would like to comment on how we can apply the stability condition (87) of the BTZ black hole in the NMG to the dual CFT whose central charge is given by [8, 24, 27]

$$c_{L/R} = \frac{3\ell}{2G} \left[1 - \frac{1}{2m^2\ell^2} \right]. \quad (88)$$

It is well known that the exact agreement is found between the quasinormal frequencies of the BTZ black hole and the location of the poles of the retarded correlation function of the corresponding perturbations in the CFT [28]. This has provided a confirmed test of the $\text{AdS}_3/\text{CFT}_2$ correspondence. If this correspondence is still valid for the NMG, one observes from (88) that the zero central charge appears at the critical point, while the unitarity of the CFT_2 would demand the bound of $m^2 > 1/2\ell^2$ due to $c_{L/R} > 0$. For the non-critical case, we find the stability condition of $M > \ell$ from (87), which turns out to be the unitarity condition for the CFT_2 . In this case, we expect that the quasinormal modes obtained here could be derived from the location of the poles of the retarded correlation function of the corresponding perturbations in the CFT_2 . On the other hand, it seems that the quasinormal modes obtained at the critical point of $M = \ell$ could be found from the retarded correlation function of the corresponding perturbations in the LCFT as was shown in the TMG [20]. Hence, it is clear that the stability condition of BTZ black hole provides the unitarity condition of the dual (L) CFT_2 .

Finally, we would like to mention that the linearized higher dimensional critical gravities were recently investigated in the AdS spacetimes [29] but their quasinormal modes are not studied in the AdS-black hole background. The non-unitarity issue of log gravity is not still resolved, indicating that the log gravity suffers from the ghost problem.

Consequently, we have performed the stability analysis of the BTZ black hole in the NMG.

It seems that the BTZ black hole is stable against the metric perturbation by computing quasinormal modes and observing two potentials. However, the stability at the critical point is not still completely proved because two s -modes [(60) and (65)] and [(62) and (67)] are not orthogonal to each other, respectively.

Acknowledgement

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Appendix I: LCFT approach to the critical point of NMG

In this appendix, we will prove the equations of motion (53) at the critical point in the BTZ black hole of the NMG using the LCFT approach. We first note that

$$L_0 y = \bar{L}_0 y = \frac{1}{2}, \quad L_1 y = \bar{L}_1 y = 0, \quad (89)$$

and

$$L_0 h_{\mu\nu}^R = ik h_{\mu\nu}^R, \quad \bar{L}_0 h_{\mu\nu}^R = 0. \quad (90)$$

Acting on the R-logarithmic mode of $h_{\mu\nu}^{R,new}$ with $L_0(\bar{L}_0)$, we have

$$L_0 h_{\mu\nu}^{R,new} = ik h_{\mu\nu}^{R,new} + \frac{1}{2} h_{\mu\nu}^R, \quad \bar{L}_0 h_{\mu\nu}^{R,new} = \frac{1}{2} h_{\mu\nu}^R, \quad (91)$$

which show that $h_{\mu\nu}^{R,new}$ is not an eigenstate of L_0 or \bar{L}_0 as like the new mode at the chiral point in AdS₃ of CTMG [6], but it could be an eigenstate of the subtraction operator $L_0 - \bar{L}_0$. The two representations of L_0 and \bar{L}_0 are now given by the Jordan cell form

$$\begin{aligned} L_0 \begin{pmatrix} h_{\mu\nu}^{R,new} \\ h_{\mu\nu}^R \end{pmatrix} &= \begin{pmatrix} ik & \frac{1}{2} \\ 0 & ik \end{pmatrix} \begin{pmatrix} h_{\mu\nu}^{R,new} \\ h_{\mu\nu}^R \end{pmatrix}, \\ \bar{L}_0 \begin{pmatrix} h_{\mu\nu}^{R,new} \\ h_{\mu\nu}^R \end{pmatrix} &= \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} h_{\mu\nu}^{R,new} \\ h_{\mu\nu}^R \end{pmatrix}. \end{aligned} \quad (92)$$

Thus, it indicates that $h_{\mu\nu}^{R,new}$ is a L-logarithmic partner of $h_{\mu\nu}^R$. This shows again $h_{\mu\nu}^{R,new}$ (the logarithmic partner of $h_{\mu\nu}^R$) and $h_{\mu\nu}^R$ are an eigenstate of the subtraction operator $L_0 - \bar{L}_0$:

$$[L_0 - \bar{L}_0] \begin{pmatrix} h_{\mu\nu}^{R,new} \\ h_{\mu\nu}^R \end{pmatrix} = ik \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h_{\mu\nu}^{R,new} \\ h_{\mu\nu}^R \end{pmatrix}. \quad (93)$$

Now, using the $SL(2, R)$ quadratic Casimir of $L^2 = \frac{1}{2}(L_1 L_{-1} + L_{-1} L_1) - L_0^2$, the equation (55) is written by

$$(D^R D^L h_{\mu\nu}^{R,new})_{\mu\nu} = -(\bar{\nabla}^2 + 2) h_{\mu\nu}^{R,new} = [2(L^2 + \bar{L}^2) + 4] h_{\mu\nu}^{R,new}. \quad (94)$$

Making use of the following equations

$$L_1 h_{\mu\nu}^{R,new} = -(2 - ik) e^u \tanh(\rho) h_{\mu\nu}^{R,new}, \quad \bar{L}_1 h_{\mu\nu}^{R,new} = 0, \quad (95)$$

we arrive at

$$(D^R D^L h_{\mu\nu}^{R,new})_{\mu\nu} = -(\bar{\nabla}^2 + 2) h_{\mu\nu}^{R,new} = -2 h_{\mu\nu}^R. \quad (96)$$

Operating $(\bar{\nabla}^2 - 2\Lambda)$ on (96) with $\Lambda = -1$ and using (23) on the R-moving mode $h_{\mu\nu}^R$, this leads to (52): $(D^R D^L)h_{\mu\nu}^{R,new} \neq 0$, $(D^R D^L)^2 h_{\mu\nu}^{R,new} = 0$. We remind the reader that $\bar{L}_1 h_{\mu\nu}^{R,new} (= 0)$ in (95) is the “chiral” highest weight condition, but not $L_1 h_{\mu\nu}^{R,new} \neq 0$ as shown in (95). This differs clearly from the CTMG at the chiral point of the AdS_3 background [6], where one should impose the “highest weight” conditions of both $L_1 h_{\mu\nu}^{R,new} = \bar{L}_1 h_{\mu\nu}^{R,new} (= 0)$ [7].

On the other hand, for the new L-logarithmic mode (56), we follow the same steps as did in the new R-moving mode. In addition to (89), we have

$$L_0 h_{\mu\nu}^L = 0, \quad \bar{L}_0 h_{\mu\nu}^L = -ik h_{\mu\nu}^L. \quad (97)$$

Therefore, the new L-logarithmic mode of $h_{\mu\nu}^{L,new}$ satisfies

$$L_0 h_{\mu\nu}^{L,new} = \frac{1}{2} h_{\mu\nu}^L, \quad \bar{L}_0 h_{\mu\nu}^{L,new} = -ik h_{\mu\nu}^{L,new} + \frac{1}{2} h_{\mu\nu}^L. \quad (98)$$

Similarly, the two representations of L_0 and \bar{L}_0 take the compact matrix forms

$$\begin{aligned} L_0 \begin{pmatrix} h_{\mu\nu}^{L,new} \\ h_{\mu\nu}^L \end{pmatrix} &= \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} h_{\mu\nu}^{L,new} \\ h_{\mu\nu}^L \end{pmatrix}, \\ \bar{L}_0 \begin{pmatrix} h_{\mu\nu}^{L,new} \\ h_{\mu\nu}^L \end{pmatrix} &= \begin{pmatrix} -ik & \frac{1}{2} \\ 0 & -ik \end{pmatrix} \begin{pmatrix} h_{\mu\nu}^{L,new} \\ h_{\mu\nu}^L \end{pmatrix}. \end{aligned} \quad (99)$$

This shows again $h_{\mu\nu}^{L,new}$ (the logarithmic partner of $h_{\mu\nu}^L$) and $h_{\mu\nu}^L$ are an eignestate of the subtraction operator $L_0 - \bar{L}_0$:

$$[L_0 - \bar{L}_0] \begin{pmatrix} h_{\mu\nu}^{L,new} \\ h_{\mu\nu}^L \end{pmatrix} = ik \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h_{\mu\nu}^{L,new} \\ h_{\mu\nu}^L \end{pmatrix}. \quad (100)$$

Now, making use of the following equations

$$L_1 h_{\mu\nu}^{L,new} = 0, \quad \bar{L}_1 h_{\mu\nu}^{L,new} = -(2 + ik)e^v \tanh(\rho) y(\tau, \rho) h_{\mu\nu}^L, \quad (101)$$

where $L_1 h_{\mu\nu}^{L,new} (= 0)$ is the “anti-chiral” highest weight condition, but not for $\bar{L}_1 h_{\mu\nu}^{L,new} \neq 0$, we finally arrive at

$$(D^R D^L h_{\mu\nu}^{L,new})_{\mu\nu} = -(\bar{\nabla}^2 + 2) h_{\mu\nu}^{L,new} = -2 h_{\mu\nu}^L, \quad (102)$$

which clearly confirms (53).

Appendix II: Full perturbation analysis in BTZ black hole background

Considering the full $h_{\mu\nu}$ components given by

$$h_{\mu\nu} = \begin{pmatrix} H_{tt}(r) & H_{tr}(r) & H_{t\phi}(r) \\ H_{rt}(r) & H_{rr}(r) & H_{r\phi}(r) \\ H_{\phi t}(r) & H_{\phi r}(r) & H_{\phi\phi}(r) \end{pmatrix} e^{\omega_h t} e^{ik\phi}, \quad (103)$$

the tensor perturbation equation of $\bar{\nabla}_{\text{BTZ}}^2 h_{\mu\nu} - \left(\frac{5\Lambda}{2} + m^2\right) h_{\mu\nu} = 0$ lead to

$$(t, t); \quad -2r^2(\mathcal{M} + \Lambda r^2)^2 H_{tt}'' - 2r(\mathcal{M}^2 - \Lambda^2 r^4) H_{tt}' - \{(2\mathcal{M}k^2 + 2\Lambda m^2 r^2) - \Lambda r^2(3\mathcal{M} - \Lambda r^2) - 2r^2\omega_h^2 + 2m^2 r^2(\mathcal{M} + \Lambda r^2)\} H_{tt} + 4\Lambda^2 r^4(\mathcal{M} + \Lambda r^2)^2 H_{rr} - 8\Lambda\omega_h r^3(\mathcal{M} + \Lambda r^2) H_{tr} = 0$$

$$(t, r); \quad 2r^3(\mathcal{M} + \Lambda r^2)^3 H_{tr}'' + 2r^2(\mathcal{M} + \Lambda r^2)^2(\mathcal{M} + 3\Lambda r^2) H_{tr}' + r(\mathcal{M} + \Lambda r^2) \{-2\mathcal{M}^2 + \Lambda r^4 \times (2m^2 - 5\Lambda) + \mathcal{M}r^2(\Lambda + 2m^2) + 2k^2(\mathcal{M} + \Lambda r^2) - 2r^2\omega_h^2\} H_{tr} + 4ik(\mathcal{M} + \Lambda r^2)^2 H_{t\phi} + 4\Lambda\omega_h r^4 H_{tt} + 4\Lambda\omega_h r^4(\mathcal{M} + \Lambda r^2)^2 H_{rr} = 0$$

$$(t, \phi); \quad 2r^2(\mathcal{M} + \Lambda r^2)^2 H_{t\phi}'' - 2r(\mathcal{M} + \Lambda r^2)^2 H_{t\phi}' + \{(\mathcal{M} + \Lambda r^2)(2k^2 + (\Lambda + 2m^2)r^2) - 2r^2\omega_h^2\} H_{t\phi} + 4ikr(\mathcal{M} + \Lambda r^2)^2 H_{tr} + 4\Lambda\omega_h r^3(\mathcal{M} + \Lambda r^2) H_{r\phi} = 0$$

$$(r, r); \quad -2r^4(\mathcal{M} + \Lambda r^2)^4 H_{rr}'' - 2r^3(\mathcal{M} + \Lambda r^2)^3(\mathcal{M} + 7\Lambda r^2) H_{rr}' - r^2(\mathcal{M} + \Lambda r^2)^2 \{-4\mathcal{M}^2 + \mathcal{M}r^2(2m^2 + 5\Lambda) + \Lambda r^4(2m^2 + 13\Lambda) + 2k^2(\mathcal{M} + \Lambda r^2) - 2\omega_h^2 r^2\} H_{rr} + 4\Lambda^2 r^6 H_{tt} + -8\Lambda\omega_h r^5(\mathcal{M} + \Lambda r^2) H_{tr} + 4(\mathcal{M} + \Lambda r^2)^3 H_{\phi\phi} - 8ikr(\mathcal{M} + \Lambda r^2)^3 H_{r\phi} = 0$$

$$(r, \phi); \quad 2r^3(\mathcal{M} + \Lambda r^2)^3 H_{r\phi}'' - 2r^2(\mathcal{M} + \Lambda r^2)^2(\mathcal{M} - 3\Lambda r^2) H_{r\phi}' + r(\mathcal{M} + \Lambda r^2) \{(\mathcal{M} + \Lambda r^2) \times (2k^2 - 6\mathcal{M}) + r^2(\mathcal{M} + \Lambda r^2)(2m^2 - 5\Lambda) - 2\omega_h^2 r^2\} H_{r\phi} + 4ikr^2(\mathcal{M} + \Lambda r^2)^3 H_{rr} + 4\Lambda\omega_h r^4 H_{t\phi} + 4ik(\mathcal{M} + \Lambda r^2)^2 H_{\phi\phi} = 0$$

$$(\phi, \phi); \quad 2r^2(\mathcal{M} + \Lambda r^2)^2 H_{\phi\phi}'' - 2r(\mathcal{M} + \Lambda r^2)(3\mathcal{M} + \Lambda r^2) H_{\phi\phi}' + \{(\mathcal{M} + \Lambda r^2)(2k^2 + 4\mathcal{M} + 2m^2 r^2 + \Lambda r^2) - 2\omega_h^2 r^2\} H_{\phi\phi} - 4r^2(\mathcal{M} + \Lambda r^2)^3 H_{rr} + 8ikr(\mathcal{M} + \Lambda r^2)^2 H_{r\phi} = 0.$$

Also the TT gauge condition of $\bar{\nabla}^\mu h_{\mu\nu} = 0$ and $h = 0$ are given by

$$\begin{aligned}
t; \quad & r^2(\mathcal{M} + \Lambda r^2)^2 H'_{tr} + r(\mathcal{M} + \Lambda r^2)(\mathcal{M} + 3\Lambda r^2)H_{tr} - ik(\mathcal{M} + \Lambda r^2)H_{t\phi} + r^2\omega_h H_{tt} = 0 \\
r; \quad & r^3(\mathcal{M} + \Lambda r^2)^3 H'_{rr} + r^2(\mathcal{M} + \Lambda r^2)^2(\mathcal{M} + 4\Lambda r^2)H_{rr} + \Lambda r^4 H_{tt} + (\mathcal{M} + \Lambda r^2)^2 H_{\phi\phi} \\
& - \omega_h r^3(\mathcal{M} + \Lambda r^2)H_{tr} - ikr(\mathcal{M} + \Lambda r^2)^2 H_{r\phi} = 0 \\
\phi; \quad & -r^2(\mathcal{M} + \Lambda r^2)^2 H'_{r\phi} - r(\mathcal{M} + \Lambda r^2)(\mathcal{M} + 3\Lambda r^2)H_{r\phi} + ik(\mathcal{M} + \Lambda r^2)H_{\phi\phi} + \omega_h r^2 H_{t\phi} = 0
\end{aligned}$$

$$r^2 H_{tt} - r^2(\mathcal{M} + \Lambda r^2)^2 H_{rr} + (\mathcal{M} + \Lambda r^2)H_{\phi\phi} = 0$$

For type I(odd) metric ansatz (73), i.e., $H_{tt} = H_{tr} = H_{rr} = H_{\phi\phi} = 0$, (t, r) equation becomes

$$4ik(\mathcal{M} + \Lambda r^2)^2 H_{t\phi} = 0,$$

and the solution is $H_{t\phi} = 0$ for $k \neq 0$. However, this corresponds to the null odd-solution because in this case, we obtain $H_{r\phi} = 0$ from (t, ϕ) equation. Therefore, the s -mode ($k = 0$) solution is only admitted for type I metric ansatz (73).

On the other hand, when focusing on (t, ϕ) equation and considering type II (even) metric ansatz (74) with $H_{t\phi} = H_{r\phi} = 0$, equation (t, ϕ) reduces to

$$4ikr(\mathcal{M} + \Lambda r^2)^2 H_{tr} = 0.$$

Unless $k = 0$, we obtain $H_{tr} = 0$. In this case, $H_{tt} = 0$ is found from t component equation of $\bar{\nabla}^\mu h_{\mu\nu} = 0$. Furthermore, we have $H_{rr} = 0$ and $H_{\phi\phi} = 0$ from (t, t) and (r, r) equations. This corresponds to the null even-solution. So we also have to restrict to the s -mode case. In other words, the s -mode case leads to type I and II metric splitting for obtaining two modes of a massive graviton.

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